HE 215 : Nuclear & Particle Physics Course

Dr. Jyothsna Rani Komaragiri

Centre for High Energy Physics, IISc

Webpage: `http://chep.iisc.ac.in/Personnel/jyothsna.html`

e-mail:
jyothsna@iisc.ac.in,
jyothsna.komaragiri@gmail.com

November 2018 Lectures



Outline I

Weak Interactions

This is chapter 9 in Griffiths.

Weak Interactions

We need to be able to do calculations with weak interaction Feynman diagrams too. Also, you may have noticed that all of the interactions like to play by the rules except the weak interaction (eg. parity).

The really striking difference between the weak interaction and QED is the mass of the propagators. The photon is massless but the $W^{\pm}/Z$ are two of the heaviest known particles:

$$\begin{aligned} M_W &= 80.385(15) GeV \\ M_z &= 91.1876(21) GeV \end{aligned}$$

(taken from PDG Particle Booklet, July 2014)

As we have seen, this makes the weak force extremely short range. It also means three polarization states.

Charged Weak Interactions Feynman Rules

The Feynman Rules are just like those for QED, except different. The propagator for massive spin-1 particles is more complicated:

$$\frac{-i(g_{\mu\nu} - q_\mu q_\nu / M^2 c^2)}{q^2 - M^2 c^2}$$

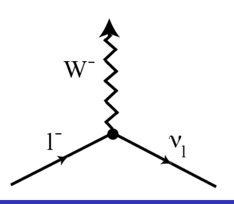
If q^2 is much smaller than $(Mc)^2$ then we can use

$$\frac{ig_{\mu\nu}}{(Mc)^2}$$

The Charged Current (CC) vertex factor is:

$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$

where $g_w = \sqrt{4\pi\alpha_w}$ is "weak coupling constant"



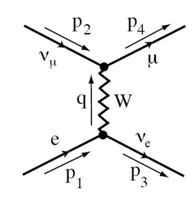
Charged Weak Interactions Feynman Rules

$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$

This strange-looking vertex factor is very important. When people talk about the $V-A$ nature of the EW interaction it is this that they are referring to.
Since we know that $\gamma^\mu\gamma^5$ is axial (remember bilinear covariants) we know that these diagrams will not conserve parity.
In real-life this will lead to very different angular distributions of decay products from weak decays than if this was not "pure" $V-A$.

Inverse Muon Decay: Electron-Neutrino scattering

Consider process $\nu_\mu + e^- \to \mu^- + \nu_e$



Let's assume that $q^2 \ll M_w^2 c^2$, so the amplitude is:

$$\mathcal{M} = \frac{g_w^2}{8(M_w c)^2}[\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)][\bar{u}(4)\gamma_\mu(1-\gamma^5)u(2)]$$

Applying Casimir's trick

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{2}\left(\frac{g_w^2}{8(M_w c)^2}\right)^2 Tr[\gamma^\mu(1-\gamma^5)(\not{p}_1 + m_e c)\gamma^\nu(1-\gamma^5)\not{p}_3] \\ &\times \quad Tr[\gamma_\mu(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma^5)(\not{p}_4 + m_\mu c)]\end{aligned}$$

Inverse Muon Decay

To evaluate these traces you need to play games with γ^5s by moving things around. For example:

$$\begin{aligned}
(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma^5) &= (1-\gamma^5)\not{p}_2(1+\gamma^5)\gamma^\nu \\
&= (1-\gamma^5)(1-\gamma^5)\not{p}_2\gamma_\nu \\
&= 2(1-\gamma^5)\not{p}_2\gamma_\nu
\end{aligned}$$

Now use trace rules for γ^5 and get

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{g_w^4}{2M_w^4}[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - (p_1\cdot p_3)g^{\mu\nu} - i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}] \\
&\times [p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - (p_2\cdot p_4)g_{\mu\nu} - i\epsilon_{\mu\nu\kappa\tau}p_2^\kappa p_4^\tau] \\
&= 2\left(\frac{g_w}{M_w c}\right)^4 (p_1\cdot p_2)(p_3\cdot p_4)
\end{aligned}$$

Inverse Muon Decay

Let's go to the CM frame (E energy of incident e^-) and neglect the electron mass:

$$
\begin{aligned}
p_1 \cdot p_2 &= [(p_1 + p_2)^2 - p_1^2 - p_2^2]/2 \\
&= \left[\frac{(2E)^2}{c^2} - 0 - 0\right]/2 = 2\left(\frac{E}{c}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
p_3 \cdot p_4 &= [(p_3 + p_4)^2 - p_3^2 - p_4^2]/2 \\
&= [(p_1 + p_2)^2 - 0 - (m_\mu c)^2]/2 \\
&= \left[\frac{(2E)^2}{c^2} - (m_\mu c)^2\right]/2 \\
&= 2\left(\frac{E}{c}\right)^2 \left[1 - \left(\frac{m_\mu c^2}{2E}\right)^2\right]
\end{aligned}
$$

Inverse Muon Decay

We end up with

$$\langle|\mathcal{M}|^2\rangle = \frac{8g_w^4E^4}{(M_Wc^2)^4}\left\{1-\left(\frac{m_\mu c^2}{2E}\right)^2\right\}$$

Applying Fermi Golden Rule to get differential scattering cross section:

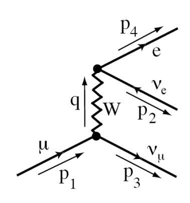
$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left[\frac{\hbar c g_w^2 E}{4\pi(M_Wc^2)^2}\right]^2\left\{1-\left(\frac{m_\mu c^2}{2E}\right)^2\right\}^2$$

and the total cross section is:

$$\sigma = \frac{1}{8\pi}\left[\frac{\hbar c g_w^2 E}{(M_Wc^2)^2}\right]^2\left\{1-\left(\frac{m_\mu c^2}{2E}\right)^2\right\}^2$$

Muon Decay

Let's do inverse-inverse muon decay:



By now, we can easily come up with an amplitude:

$$\mathcal{M} = \frac{g_w^2}{8(M_W c)^2}[\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)][\bar{u}(4)\gamma_\mu(1-\gamma^5)v(2)]$$

Which gives us something which is also familiar:

$$\langle|\mathcal{M}|^2\rangle = 2\left(\frac{g_w}{M_w c}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4)$$

Muon Decay Kinematics

$$\langle|\mathcal{M}|^2\rangle = 2\left(\frac{g_w}{M_W c}\right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4)$$

In the muon rest frame we have $p_1 = (m_\mu c, \mathbf{0})$ and

$$p_1 \cdot p_2 = m_\mu E_2$$

How about $(p_3 \cdot p_4)$? Well, this is a decay so $p_1 = p_2 + p_3 + p_4$

$$\begin{aligned}
(p_3 + p_4)^2 &= p_3^2 + p_4^2 + 2p_3 \cdot p_4 = m_e^2 c^2 + 2p_3 \cdot p_4 \\
&= (p_1 - p_2)^2 = p_1^2 + p_2^2 - 2p_1 \cdot p_2 = m_\mu^2 c^2 - 2p_1 \cdot p_2 \\
p_3 \cdot p_4 &= \frac{(m_\mu^2 - m_e^2)c^2}{2} - m_\mu E_2
\end{aligned}$$

Muon Decay - Fermi's Golden Rule

From now on let's ignore the electron mass giving

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{g_w}{M_w c}\right)^4 m_\mu^2 E_2(m_\mu c^2 - 2E_2)$$

This is a 3-body decay for which we can use Fermi's Golden Rule

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{2\hbar m_\mu}\left(\frac{d^3\boldsymbol{p}_2}{(2\pi)^3 2|\boldsymbol{p_2}|}\right)\left(\frac{d^3\boldsymbol{p}_3}{(2\pi)^3 2|\boldsymbol{p_3}|}\right)\left(\frac{d^3\boldsymbol{p}_4}{(2\pi)^3 2|\boldsymbol{p_4}|}\right)$$

$$\times(2\pi)^4\delta^4(p_1 - p_2 - p_3 - p_4)$$

Splitting the delta function:

$$\delta^4(p_1 - p_2 - p_3 - p_4) = \delta(m_\mu c - |\boldsymbol{p_2}| - |\boldsymbol{p_3}| - |\boldsymbol{p_4}|)\delta^3(|\boldsymbol{p_2}| + |\boldsymbol{p_3}| + |\boldsymbol{p_4}|)$$

Muon Decay - Fermi's Golden Rule

So, we have seen expressions like this before. We know already that $\mathcal{M}$ depends on E_2 so we should not try the p_2 integral first. In the end we will want to express our result in terms of electron energy (the observable) so we should also delay integrating over p_4. That leaves p_3. Do it by splitting the delta function and get:

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{16(2\pi)^5\hbar m_\mu} \frac{(d^3\boldsymbol{p}_2)(d^3\boldsymbol{p}_4)}{|\boldsymbol{p_2}||\boldsymbol{p_2}+\boldsymbol{p_4}||\boldsymbol{p_4}|} \delta(m_\mu c - |\boldsymbol{p_2}| - |\boldsymbol{p_2}+\boldsymbol{p_4}| - |\boldsymbol{p_4}|)$$

The delta we used to get rid of p_3 enforces $|\boldsymbol{p_3}| = |\boldsymbol{p_2}+\boldsymbol{p_4}|$

Muon Decay - Changing Variables

Next we will do the $\boldsymbol{p_2}$ integral. In spherical coordinates we have

$$d^3\boldsymbol{p}_2 = |\boldsymbol{p}_2|^2 d\boldsymbol{p}_2 d\Omega = |\boldsymbol{p}_2|^2 d\boldsymbol{p}_2 \sin\theta d\theta d\phi$$

The ϕ integral gives just 2π and the θ integral is nasty. What is θ?
We choose our polar axis to be along the $\boldsymbol{p}_4$ direction and θ is then the angle between the electron and the neutrino (particles 4 and 2).
We should also remember what we know about $\boldsymbol{p_3}$:

$$|\boldsymbol{p_2} + \boldsymbol{p_4}|^2 = |\boldsymbol{p_2}|^2 + |\boldsymbol{p_4}|^2 + 2|\boldsymbol{p_2}||\boldsymbol{p_4}|\cos\theta \equiv u^2$$

To carry out θ integration we change variables $\theta \to u$

$$2udu = -2|\boldsymbol{p_2}||\boldsymbol{p_4}|\sin\theta d\theta$$

Muon Decay - Changing Variables

So:

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{16(2\pi)^5\hbar m_\mu}\frac{d^3\boldsymbol{p}_4}{|\boldsymbol{p}_4|^2}d\boldsymbol{p}_2\int_{u_-}^{u_+}\delta(m_\mu c - |\boldsymbol{p}_2| - |\boldsymbol{p}_4| - u)$$

where

$$u_\pm \equiv \sqrt{|\boldsymbol{p}_2|^2 + |\boldsymbol{p}_4|^2 \pm 2|\boldsymbol{p}_2||\boldsymbol{p}_4|} = |\,|\boldsymbol{p}_2| \pm |\boldsymbol{p}_4|\,|$$

The u integral is 1 if:

$$u_- < m_\mu c - |\boldsymbol{p}_2| - |\boldsymbol{p}_4| < u_+$$

and 0 otherwise.

Muon Decay - Changing Variables

This inequality can be written as:

$$\left|\, |\boldsymbol{p_2}| - |\boldsymbol{p_4}| \,\right| < m_\mu c - |\boldsymbol{p_2}| - |\boldsymbol{p_4}| < \left|\, |\boldsymbol{p_2}| + |\boldsymbol{p_4}| \,\right|$$

This is equivalent to 3 inequalities

$$\left\{ \begin{array}{c} |\boldsymbol{p_2}| < \frac{1}{2} m_\mu c \\ |\boldsymbol{p_4}| < \frac{1}{2} m_\mu c \\ (|\boldsymbol{p_2}| + |\boldsymbol{p_4}|) > \frac{1}{2} m_\mu c \end{array} \right\}$$

This matches our intuition: the maximum energy for any one particle is half of the total which is the same as the minimum energy for any combination of two particles.

These inequalities specify the limits:
$|\boldsymbol{p_2}|$ runs from $\frac{1}{2} m_\mu c - |\boldsymbol{p_4}|$ to $\frac{1}{2} m_\mu c$
and
$|\boldsymbol{p_4}|$ runs from 0 to $\frac{1}{2} m_\mu c$

Muon Decay - Changing Variables

The θ and ϕ integrals leave us with

$$d\Gamma = \frac{\langle |\mathcal{M}|^2 \rangle}{(4\pi)^4 \hbar m_\mu} d|\mathbf{p_2}| \frac{d^3\mathbf{p}_4}{|\mathbf{p_4}|^2}$$

Now we have to put in $\mathcal{M}$ so that we can finally evaluate the p_2 integral

$$\begin{aligned} d\Gamma &= \left(\frac{g_w}{4\pi M_w}\right)^4 \frac{m_\mu}{\hbar c^2} \frac{d^3\mathbf{p}_4}{|\mathbf{p_4}|^2} \int_{(1/2)m_\mu c - |\mathbf{p_4}|}^{(1/2)m_\mu c} |\mathbf{p_2}|(m_\mu c - 2|\mathbf{p_2}|) d|\mathbf{p_2}| \\ &= \left(\frac{g_w}{4\pi M_w c}\right)^4 \frac{m_\mu}{\hbar c^2} \left(\frac{m_\mu c}{2} - \frac{2}{3}|\mathbf{p_4}|\right) d^3\mathbf{p}_4 \end{aligned}$$

OK, at last we have just one integral left!

Muon Decay - The p_4 Integral

So, we now only have to do the p_4 integral. Writing

$$d^3\boldsymbol{p}_4 = 4\pi|\boldsymbol{p}_4|^2 d|\boldsymbol{p}_4|$$

we can write the differential rate as (with $E = |\boldsymbol{p_4}c|$):

$$\frac{d\Gamma}{dE} = \left(\frac{g_w}{M_w c}\right)^4 \frac{m_\mu^2 E^2}{2\hbar(4\pi)^3}\left(1 - \frac{4E}{3m_\mu c^2}\right)$$

This shows us the rate as a function of the electron energy.

Muon Decay - Differential Rate

$$\frac{d\Gamma}{dE} \sim E^2\left(1 - \frac{4E}{3m_\mu c^2}\right) \qquad\qquad E \leqslant \frac{1}{2}m_\mu c^2$$

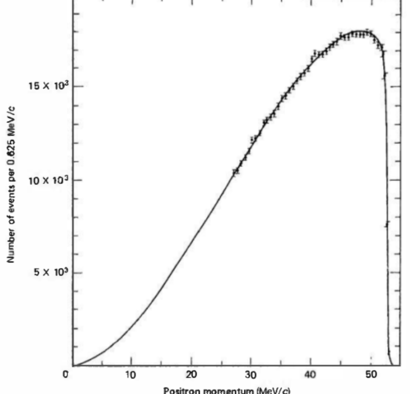


Fig. 9.1 Experimental spectrum of positrons in $\mu^+ \to e^+ + \nu_e + \nu_\mu$. The solid line is the theoretically predicted spectrum based on Equation (9.33), corrected for electromagnetic effects. (*Source*: Bardon, M. *et al.* (1965) *Physical Review Letters*, 14, 449. For the latest high-precision data on muon decay go to the TWIST collaboration web site at TRIUMF, Vancouver, BC.)

Muon Decay - Fermi Coupling

Now we can integrate over electron energy (see text) and derive an expression for the lifetime of the muon:

$$\tau = \frac{1}{\Gamma} = \left(\frac{M_w}{m_\mu g_w}\right)^4 \frac{12\hbar(8\pi)^3}{m_\mu c^2}$$

Note that this expression contains the ratio of g_w and M_w. It is traditional to express weak interaction formulas in terms of the “Fermi Coupling Constant”

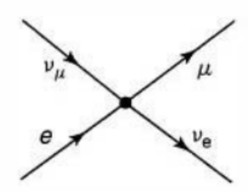
$$G_F \equiv \frac{\sqrt{2}}{8}\left(\frac{g_w}{M_w c^2}\right)^2 (\hbar c)^3$$

So, the muon lifetime could be written as

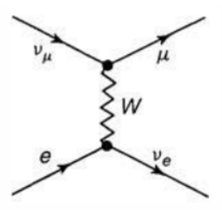
$$\tau = \frac{192\pi^3\hbar^7}{G_F^2 m_\mu^5 c^4}$$

Muon Decay - Fermi Coupling

In Fermi's original theory of beta decay (1933) there was no W; the interaction was supposed to be a direct four particle coupling, represented in the Feynman language by a diagram of the form:



From the modern perspective, Fermi's theory combined the W propagator with the two vertex factors, (to make an effective four-particle coupling constant G_F) in the diagram:



Muon Decay - Strength of the Weak Force

Given the observed muon lifetime τ, the muon mass and the W mass we can solve for g_w and get

$$g_w = 0.66$$

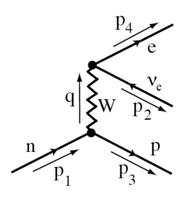
This leads to a calculation of the intrinsic strength of the weak interaction:

$$\alpha_w = \frac{g_w^2}{4\pi} = \frac{1}{29}$$

Significantly higher than the α of QED!!! The intrinsic coupling in the weak interaction is stronger than EM, only the propagators are more massive.

Neutron Decay

We are not going to do neutron decay in any detail, you can see that in Griffiths. However, I'll pick out a few interesting points. We could attempt to treat neutron decay as if neutrons and protons were elementary particles and just see what happens to our lifetime calculation.



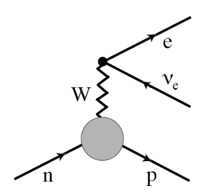
One difference between this decay and muon decay is that all of the outgoing particles in muon decay were essentially massless while no masses can be ignored here. The proton mass is very similar to the neutron mass and the electron mass (0.5 MeV) is a significant fraction of the proton-neutron mass difference (1.3 MeV).

Neutron Decay

If we were to treat the neutron this way we would get the wrong answer for the lifetime:

$$\tau_{th} = 1318s,\ \tau_{exp} = 886s$$

It is not OK to treat the neutron like a point particle.



In addition to introducing an unknown “blob” we modify the charged weak vertex because of all the strong interaction “stuff” going on inside the neutron:

$$\frac{-g_W}{2\sqrt{2}}\gamma^{\mu}(c_V - c_A\gamma^5)$$

where c_v is the correction to the vector 'weak charge', and c_A is the correction to the axial vector 'weak charge'.

Neutron Decay

We measure:

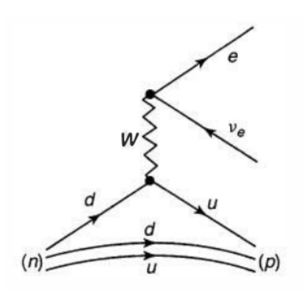
$$c_V = 1.000 \pm 0.003, \qquad c_A = 1.270 \pm 0.003$$

The vector weak charge is not modified by the strong interactions within the nucleon. Presumably, like electric charge, it is 'protected' by a conservation law; we call this the 'Conserved Vector Current' (CVC) hypothesis.
Even the axial term is not altered much; evidently it is 'almost' conserved. We call this the 'Partially Conserved Axial Current' (PCAC) hypothesis.

With this modification we predict a neutron lifetime of 901s, compatible with experiment.

Neutron Decay

There is yet another correction to be made. The underlying quark process here is $d \rightarrow u + W$ (with two spectators):

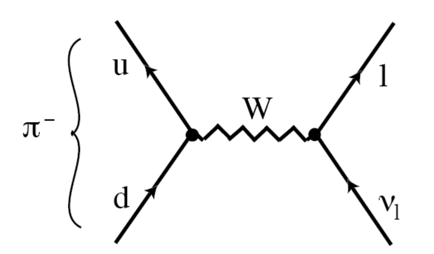


and this quark vertex carries a factor of $\cos\theta_c$, where $\theta_c = 13.15$ is the "Cabibbo angle".

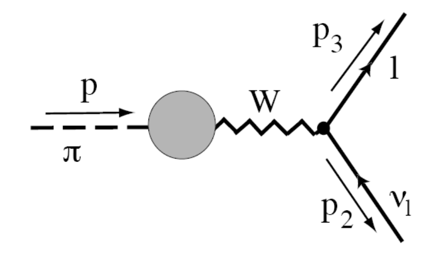
With this modification theoretical value for the neutron lifetime is 950s.

Pion Decay

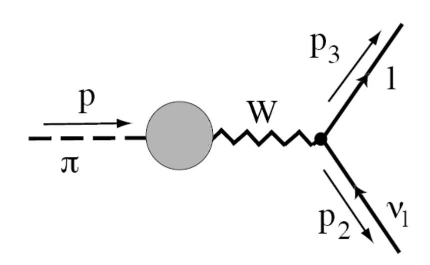
Pion decay is also a weak decay of a composite object which you could view as



However, we would run into trouble taking that route so let's put in the "blob" right from the beginning:



Pion Decay



We know how to use Feynman rules for part of this diagram:

$$\mathcal{M} = \frac{g_w^2}{8(M_W c)^2}[\bar{u}(3)\gamma_\mu(1-\gamma^5)v(2)]F^\mu$$

F^μ is a “form factor” (a lack-of-knowledge-hider). The form factor is some constant times the momentum 4-vector:

$$F^\mu = f_\pi p^\mu$$

f_π is called the “pion decay constant” (experimentally $f_\pi \simeq 93MeV$)

Pion Decay

Summing over spins and invoking Casimir's trick we have:

$$\langle|\mathcal{M}|^2\rangle = \left[\frac{f_\pi}{8}\left(\frac{g_w}{M_w c}\right)^2\right]^2 p_\mu p_\nu Tr[\gamma^\mu(1-\gamma^5)\not{p}_2\gamma^\nu(1-\gamma^5)(\not{p}_3 + m_l c)]$$

and using trace theorems we get

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{8}\left[f_\pi\left(\frac{g_w}{M_w c}\right)^2\right]^2 [2(p\cdot p_2)(p\cdot p_3) - p^2(p_2\cdot p_3)]$$

Pion Decay

Using momentum conservation $p = p_2 + p_3$ we have (remember that particle 2 is a neutrino, so $p_2^2 = 0$):

$$\begin{aligned} p \cdot p_2 &= p_2 \cdot p_3 \\ p \cdot p_3 &= m_l^2 c^2 + p_2 \cdot p_3 \\ p^2 &= p_2^2 + p_3^2 + 2 p_2 \cdot p_3 \\ 2 p_2 \cdot p_3 &= (m_\pi^2 - m_l^2) c^2 \end{aligned}$$

So, finally

$$\langle |\mathcal{M}|^2 \rangle = \left(\frac{g_w}{2 M_w} \right)^4 f_\pi^2 m_l^2 (m_\pi^2 - m_l^2)$$

So, it is just a constant!

Pion Decay

Plugging this into the decay rate formula gives

$$\Gamma = \frac{|\boldsymbol{p}_2|}{8\pi\hbar m_\pi^2 c}\langle|\mathcal{M}|^2\rangle$$

We could show

$$|\boldsymbol{p}_2| = \frac{c}{2m_\pi}(m_\pi^2 - m_l^2)$$

Plugging in:

$$\Gamma = \frac{f_\pi^2}{\pi\hbar m_\pi^3}\left(\frac{g_w}{4M_w}\right)^4 m_l^2(m_\pi^2 - m_l^2)^2$$

Pion Decay

This formula:

$$\Gamma = \frac{f_\pi^2}{\pi\hbar m_\pi^3}\left(\frac{g_w}{4M_w}\right)^4 m_l^2(m_\pi^2 - m_l^2)^2$$

means that we have to know f_π in order to calculate the pion lifetime. However, we can make that unknown cancel out if we want to predict the ratio of rates (branching ratio):

$$\frac{\Gamma(\pi_- \to e^- + \bar{\nu}_e)}{\Gamma(\pi_- \to \mu_- + \bar{\nu}_\mu)} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} = 1.28 \times 10^{-4}$$

This agrees very well with the experimental value: 1.23×10^{-4}.

Pion Decay

Why does a π_- prefer to decay to a muon instead of an electron. If I ignore mass differences I expect that muons and electrons are the same thing. If I include them I expect electrons to be preferred because they are much lighter and so the decay has more phase-space.

The pion is spin-0, so the electron and the anti-neutrino must emerge with opposite spins. We know that the anti-neutrino is right-handed, so the electron is as well. If electrons were massless then right-handed electrons would never happen. Looking at Γ equation, the decay $\pi^- \rightarrow e^- + \bar{\nu}_e$ would not happen if electron is massless. As the physical electron is very close to being massless the decay is suppressed.

Charged Weak Interactions of Quarks

Remember the funny behaviours of the weak interaction. For leptons we have a generational structure:

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$$

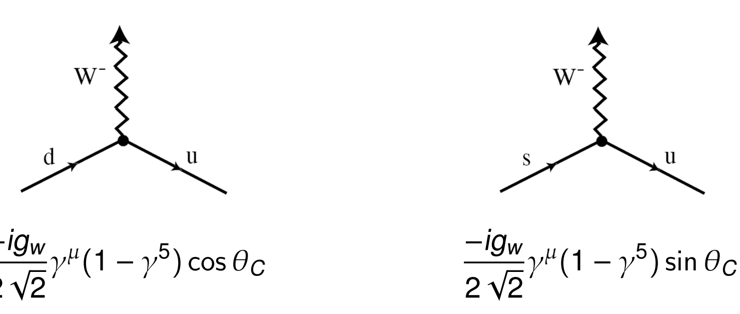
and the weak interaction changes a particle within its generation. For quarks the structure is similar:

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}$$

but the weak force can transform one quark to another inside or outside of its generation. There is no conservation of “upness” or “downness” or “upness+downness”.Otherwise, the lightest B meson would be stable, as would the lightest strange particle.

Charged Weak Interactions of Quarks

Ultimately, of course, we are looking for a set of Feynman rules which allow us to describe the charged weak interactions of quarks. We will need a new vertex factor for a quark-quark-W vertex. For a model with only three quarks (u, d, s) Cabibbo suggested:



$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\cos\theta_C \qquad \frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\sin\theta_C$$

The only difference between a transformation within a generation and without is changing the factor from cosine to sine.

Charged Weak Interactions of Quarks

Even in the weak interaction an intra-generational change is much more likely than an inter-generational change, so evidently the angle is not 45° but rather something small.

It is measured to be approximately:

$$\theta_C = 13.15^\circ$$

Example 9.2

We have done the decay $\pi_- \to l + \nu$ in some detail. This involved $d + \bar{u} \to W^-$. What about a decay which crosses generations? $K^- \to l^- + \bar{\nu}$ is instead $s + \bar{u} \to W^-$. Just what we need!

By analogy to pion decay we have:

$$\Gamma = \frac{f_K^2}{\pi\hbar m_K^3}\left(\frac{g_w}{4M_w}\right)^4 m_l^2(m_K^2 - m_l^2)^2$$

Charged Weak Interactions of Quarks

$$\Gamma = \frac{f_K^2}{\pi \hbar m_K^3} \left(\frac{g_w}{4 M_w} \right)^4 m_l^2 (m_K^2 - m_l^2)^2$$

All we have done is changed f_π to f_K. According to Cabibbo the only difference between them is that f_π has a factor of $\cos\theta_C$ inside while f_K has $\sin\theta_C$. Therefore:

$$\frac{\Gamma(K^- \to l^- + \bar{\nu})}{\Gamma(\pi^- \to l^- + \bar{\nu})} = \tan^2\theta_C \left(\frac{m_\pi}{m_K} \right)^3 \left(\frac{m_K^2 - m_l^2}{m_\pi^2 - m_l^2} \right)^2$$

These are called leptonic decays.

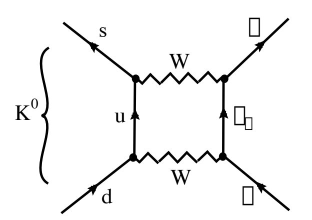
Putting in the numbers, the above ratio is 0.96 for the muon mode ($l = \mu$) and 0.19 for the electron mode ($l = e$). The observed ratios are 1.34 and 0.26 respectively.Perfect agreement is not to be expected as these decays are pure axial vector.

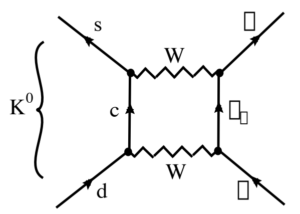
Charged Weak Interactions of Quarks

Cabibbo did a good job. However, there was a problem. According to Cabibbo, the interaction:

$$K^0 \to \mu^+\mu^-$$

is an allowed process. Our calculation would have it happening far more often then is observed. What are we missing??





The diagram on the left has an amplitude proportional to $\sin\theta_C \cos\theta_C$
The diagram on the right is introduced to partially cancel the one on the left by giving it an amplitude proportional to $-\sin\theta_C \cos\theta_C$.

Charged Weak Interactions of Quarks

This is the GIM mechanism. Convenient for theorists I guess, but it does introduce a whole new fundamental particle just to make this decay rate work out better... A "charm" quark....

Cabibbo + GIM has then given us a nice model of charged weak interactions of quarks. However, what does this mean?? Why do we have this Cabibbo angle?

Charged Weak Interactions of Quarks

We say that the weak force does not couple to the physical quarks in the way that it couples to the physical leptons. Instead of using the physical quark states we need to introduce weak eigenstates of quarks which look like

$$\begin{aligned} d' &= d\cos\theta_C + s\sin\theta_C \\ s' &= -d\sin\theta_C + s\cos\theta_C \end{aligned}$$

We can also write this in matrix form:

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

Charged Weak Interactions of Quarks

So, the W does not couple to the quarks we observe but rather to the Cabibbo-rotated states

$$\begin{pmatrix} u \\ d' \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix}$$

in exactly the same way that they couple to lepton pairs! If you want to express their couplings to physical particles you need to write

$$\begin{pmatrix} u \\ d' \end{pmatrix} = \begin{pmatrix} u \\ d \cos \theta_C + s \sin \theta_C \end{pmatrix}, \begin{pmatrix} c \\ s' \end{pmatrix} = \begin{pmatrix} c \\ -d \sin \theta_C + s \cos \theta_C \end{pmatrix}$$

CKM

Of course, the charm quark **does** exist and so do some others. Kobayashi and Maskawa had generalized the Cabibbo-GIM scheme to handle three generations of quarks:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

These nine elements are not all independent of each other. they can be written in terms of 3 Cabibbo-like angles and a phase:

$$V = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix}$$

V is Cabbibo-Kobayashi-Maskawa (CKM) matrix.
The values of all of these elements are taken from experiment.

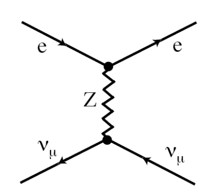
Neutral Weak Interactions

So far we have discussed the charged weak interaction (interactions with $W^{\pm}$) but we know there is another weak boson to consider. But why?

- In the 1960s there was no compelling experimental evidence for neutral weak interactions.
- Fermi's original model of weak interactions suggested charged weak interactions but did not lead to neutral ones.
- Why invent a particle (the Z boson) without theoretical or experimental justification?
- Well, there is a pattern here which should be starting to get familiar to you....

Neutral Weak Interactions

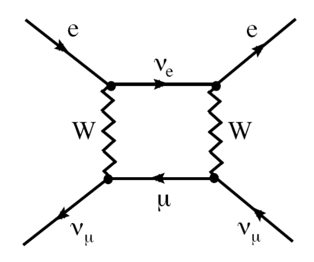
Some theorists (Glashow, Weinberg, Salam) wanted to do something cool, namely unify the EM and weak interaction. Their theory required the existence of neutral weak interactions and a Z boson. However, this would prove to be very difficult to observe! At “normal” energies the photon-exchange diagram would totally dominate over the Z exchange diagram. Needed to look in neutrino interactions to get a clean signal



First evidence in Gargamelle in 1973.

Neutral Weak Interactions

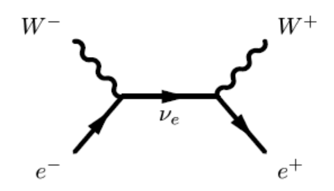
Of course, you can do this kind of scattering in charged weak interactions at higher orders:



However, these higher order diagrams give much too small a cross section to account for what Gargamelle saw.

Do we Need a Z?

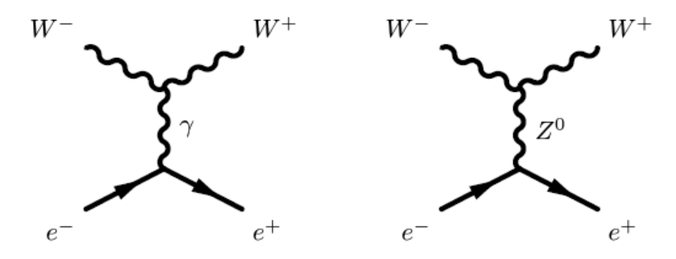
It turns out that one reason we need a Z is to solve a theoretical puzzle. If you calculate the cross section of $e^+ + e^- \rightarrow W^+ + W^-$ (time is up) it will give an answer which is too big (violates unitarity bound) if the only diagram is:



....so there must be some other diagrams out there which fix this...

Do we Need a Z?

So, we require two additional contributions (time is up):



The Weak Mixing Angle

There is a relationship between the charged and neutral weak interaction. For example, if you know the mass of the W and wish to calculate the mass of the Z you can write:

$$M_w = M_z \cos\theta_w$$

where θ_w is the weak mixing angle. Also known as “Weinberg angle”.

Experimentally:

$$\sin^2\theta_w = 0.2314 \rightarrow \theta_w = 28.75$$

What About Coupling Constants?

The vertex factor for Z interactions will involve a coupling g_z. We can write that in terms of the strength of the charged weak interaction via:

$$g_z = g_w \cos\theta_w$$

Better still, let's relate those couplings to the EM coupling

$$g_w = \frac{g_e}{\sin\theta_w}, \; g_z = \frac{g_e \cos\theta_w}{\sin\theta_w}$$

Again we see that the Weak interaction intrinsic strength is greater than that of the EM interaction.

Feynman Rules for the Neutral Weak Interaction

The propagator for the Z is just like that for the W

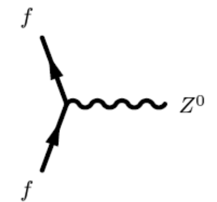
$$\frac{-i\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M_Z^2 c^2}\right)}{q^2 - M_Z^2 c^2}$$

In the case that $q^2 \ll M_Z^2 c^2$ it reduces to

$$\frac{ig_{\mu\nu}}{(M_Z c)^2}$$

However the vertex factor is different

$$\frac{-ig_z}{2}\gamma^\mu(c_V^f - c_A^f\gamma^5)$$



The Z vertex does not change the flavour of the fermion (no FCNC).

Feynman Rules for the Neutral Weak Interaction

Vertex factor:

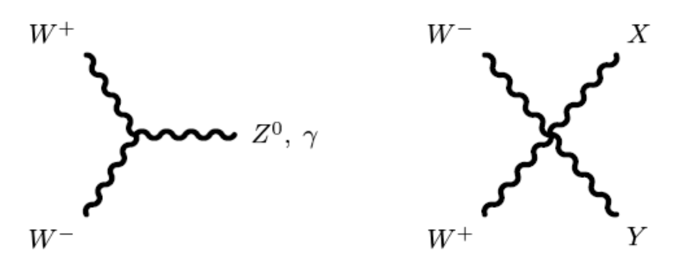
$$\frac{-ig_z}{2}\gamma^{\mu}(c_V^f - c_A^f\gamma^5)$$

The vector and axial couplings are different for each particle:

Particle	c_V	c_A
ν_l	$1/2$	$1/2$
l	$-\frac{1}{2} + 2\sin^2\theta_w$	$-1/2$
u,c,t	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_w$	$1/2$
d,s,b	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_w$	$-1/2$

Gauge Boson Self-Coupling

Note: like gluons in QCD and unlike photons in QED the bosons of the weak interaction carry “weak charge” and so self-couple. Have a look at all the funny vertices in Appendix D in Griffiths. For example:



where X and Y can be (γ, γ), (γ, Z), (Z, Z), (W^+W^-).

γ vs. Z^0

- The Z^0 couples to every charged fermion, just like the photon does.

$$Z/\gamma \rightarrow f\,\bar{f}$$

This made it difficult to detect the Z^0 because at low energies, the QED effects dominate. Nevertheless, there are always small weak effects in otherwise electromagnetic systems (e.g. atomic parity violation).

- Unlike the photon, the Z^0 also couples to neutrinos.

$$Z \rightarrow \nu\,\bar{\nu}$$

Neutrino experiments are never easy, but at least they allow us to isolate the weak interaction.

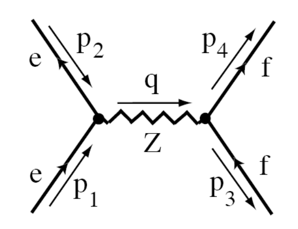
- An alternative is to work at high energy - specifically, in the neighborhood of the Z^0 mass, where the denominator of the Z^0 propagator is small, and the 'weak interaction' is large.

Electron-Positron Scattering Near the Z Pole

Consider the process:

$$e^+ + e^- \to f + \bar{f}$$

where f is any quark or lepton.



The amplitude is

$$\mathcal{M} = -\frac{g_z^2}{4(q^2 - (M_z c)^2)}[\bar{u}(4)\gamma^\mu(c_V^f - c_A^f\gamma^5)v(3)] \times\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{(M_z c)^2}\right)[\bar{v}(2)\gamma^\nu(c_V^e - c_A^e\gamma^5)u(1)]$$

Electron-Positron Scattering Near the Z Pole

$$\begin{aligned}\mathcal{M} &= -\frac{g_z^2}{4(q^2-(M_z c)^2)}[\bar{u}(4)\gamma^\mu(c_V^f - c_A^f\gamma^5)v(3)] \\ &\quad \times\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{(M_z c)^2}\right)[\bar{v}(2)\gamma^\nu(c_V^e - c_A^e\gamma^5)u(1)]\end{aligned}$$

where $q = p_1 + p_2 = p_3 + p_4$. We can ignore the lepton and quark masses since they are insignificant compared to 90GeV. The second term in the propagator

$$\frac{q_\mu q_\nu}{(M_z c)^2}$$

contributes nothing. q_μ contracts with γ^μ and writing q as $p_1 + p_2$ or $p_3 + p_4$ and it gives combinations like $\bar{u}(4)\not{p}_4$ and $\not{p}_3 v(3)$. Using the Dirac equation they equal $\bar{u}(4)m_4$ and $-m_3 v(3)$ and we are ignoring mass....

Electron-Positron Scattering Near the Z Pole

So, now we have

$$\mathcal{M} = -\frac{g_z^2}{4[q^2 - (M_Z c)^2]}[\bar{u}(4)\gamma^\mu(c_V^f - c_A^f\gamma^5)v(3)]\times[\bar{v}(2)\gamma_\mu(c_V^e - c_A^e\gamma^5)u(1)]$$

Doing the usual summing and averaging:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \left[\frac{g_z^2}{8(q^2 - (M_z c)^2)}\right]^2 Tr(\gamma^\mu(c_V^f - c_A^f\gamma^5)\not{p}_3\gamma^\nu(c_V^f - c_A^f\gamma^5)\not{p}_4) \\ &\quad \times Tr(\gamma_\mu(c_V^e - c_A^e\gamma^5)\not{p}_1\gamma_\nu(c_V^e - c_A^e\gamma^5)\not{p}_2)\end{aligned}$$

Now we need to evaulate the traces.

Electron-Positron Scattering Near the Z Pole

$$Tr(\gamma^\mu(c_V^f - c_A^f\gamma^5)\not{p}_3\gamma^\nu(c_V^f - c_A^f\gamma^5)\not{p}_4)$$

The traces are easiest to evaluate by first moving the c_V and c_A terms together:

$$\begin{aligned} & \gamma^\mu(c_V - c_A\gamma^5)\not{p}_3\gamma^\nu(c_V - c_A\gamma^5)\not{p}_4 \\ = \; & \gamma^\mu(c_V - c_A\gamma^5)^2\not{p}_3\gamma^\nu\not{p}_4 \\ = \; & \gamma^\mu(c_V^2 + c_A^2)\not{p}_3\gamma^\nu\not{p}_4 - 2c_Vc_A\gamma^\mu\gamma^5\not{p}_3\gamma^\nu\not{p}_4 \end{aligned}$$

Each of these terms can be evaluated using trace theorems. The second trace is a lot like the first one.

Electron-Positron Scattering Near the Z Pole

In the CM frame we get:

$$\langle|\mathcal{M}|^2\rangle = \left[\frac{g_z^2 E^2}{(2E)^2 - (M_z c^2)^2}\right]^2$$

$$\times\left\{[(c_V^f)^2 + (c_A^f)^2][(c_V^e)^2 + (c_A^e)^2](1 + \cos^2\theta) - 8c_V^f c_A^f c_V^e c_A^e \cos\theta\right\}$$

Plug this into the Fermi Golden Rule for scattering and integrate to get the total cross section

$$\sigma = \frac{1}{3\pi}\left(\frac{\hbar c g_z^2 E}{4[(2E)^2 - (M_z c^2)^2]}\right)^2 [(c_V^f)^2 + (c_A^f)^2][(c_V^e)^2 + (c_A^e)^2]$$

E is the energy of either the electron or the positron. So, $2E$ is the total COM energy of the system. That means that when the COM energy is exactly at the Z threshold this cross section is infinite. That is probably not correct....

Electron-Positron Scattering Near the Z Pole

The source of our problem is that the Z is not a stable particle. We need to modify the Z propagator to account for its finite lifetime:

- Recall the wavefunction of a stable particle ($c = \hbar = 1$)

$$\psi(\boldsymbol{r}, t) = \psi(\boldsymbol{r})e^{-iEt}$$

- Since the particle is stable the probability of finding the particle somewhere is 1, so the wavefunction is normalized:

$$P(t) = \int |\psi|^2 d^3 r = 1$$

- If the particle is unstable, we expect the probability to fall off with time according to the decay rate Γ

$$P(t) = \int |\psi|^2 d^3 r = e^{-\Gamma t}$$

Electron-Positron Scattering Near the Z Pole

- In the particle rest frame, this means that

$$\psi(\boldsymbol{r}, t) = \psi(\boldsymbol{r}) e^{-iMt - \Gamma t/2}$$

- Then apply the substitution $M \to M - \frac{i\Gamma}{2}$ to the Z propagator. Then assume any Γ^2 terms can be neglected

$$\frac{1}{q^2 - M^2} \to \frac{1}{q^2 - (M - i\Gamma/2)^2} \simeq \frac{1}{q^2 - M^2 + iM\Gamma}$$

The cross section then takes the form (putting back the c's and $\hbar$s)

$$\sigma \sim \frac{1}{[(2E)^2 - (M_z c^2)^2]^2 + (\hbar M_z c^2 \Gamma_z)^2}$$

This is called a Breit-Wigner resonance. Both the height and width of the peak are determined by the decay rate.

Electron-Positron Scattering Near the Z Pole

At low energies the EM process dominates $e^+e^- \to f\bar{f}$

$$\frac{\sigma_z}{\sigma_\gamma} \simeq 2\left(\frac{E}{M_z c^2}\right)^4$$

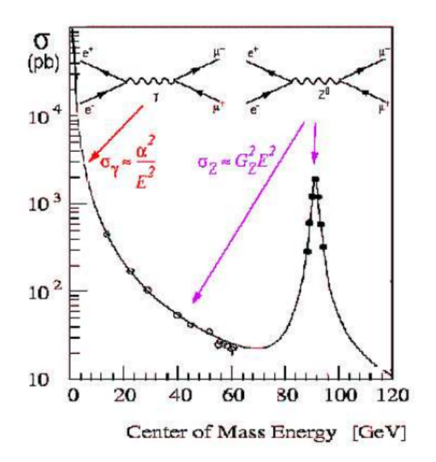
For instance, at $2E = (1/2)M_z c^2$ the weak contribution is less than 1%

However, near the Z peak ($2E = M_z c^2$) we have

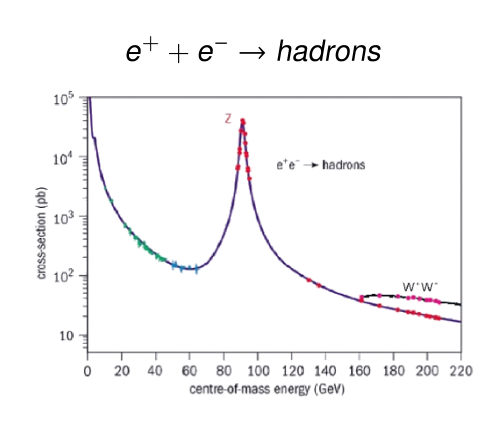
$$\frac{\sigma_z}{\sigma_\gamma} \simeq \frac{1}{8}\left(\frac{M_z c^2}{\hbar\Gamma_z}\right) \simeq 200$$

Electron-Positron Scattering Near the Z Pole

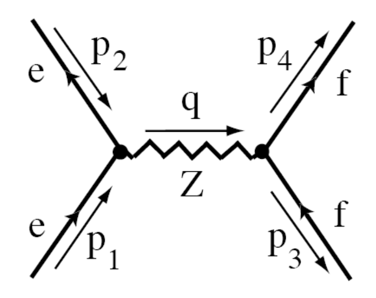
$$e^+ + e^- \to \mu^+ + \mu^-$$



$$e^+ + e^- \to hadrons$$



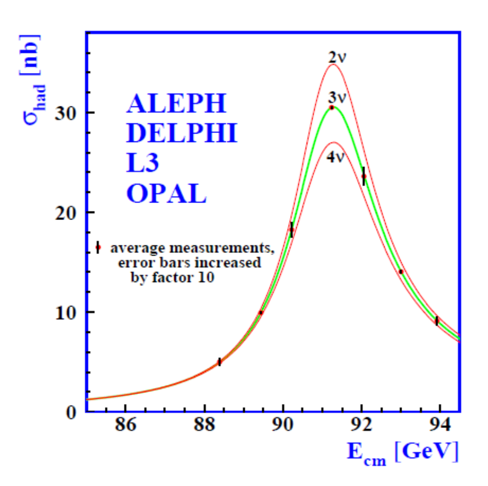
Electron-Positron Scattering Near the Z Pole



LEP did a LOT of this. It is an extremely well understood process which has yielded precision measurements of many aspects of the SM.
LEP was already planned and approved before the W/Z were discovered.
Carlo Rubbia sort-of “slipped-in” a proton-antiproton collider and the UA1/UA2 while they were waiting...
W/Z were discovered at UA1/UA2 experiments.

Z Width

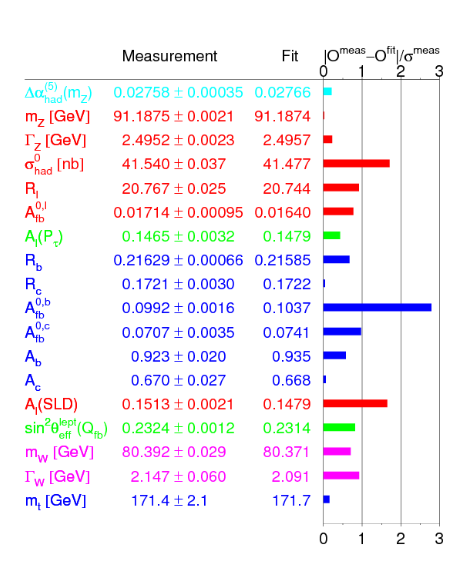
Γ_Z can be calculated in the Standard Model by putting Z^0 in the initial state. When this is done it is found that there cannot be a 4th lepton generation with a light neutrino.



An aside - The Precision of LEP.....from Wikipedia

Dr. Bagger on precision and the mass of the Z boson at CERN: ”The experimenters found that the Z boson got heavier at certain times of the day. This was a very high-precision experiment. They discovered that the patterns of the particle getting heavier corresponded to the tides. The gravitational adjustments due to tides slightly changed the shape of the collider over the course of the day.
After adjusting for tidal effects, they found that the Z boson was heavier in spring and lighter in fall. This was because there's a lake in Geneva near the detector, that is drained in Fall to make room for the spring snow-melt. So the bigger lake in the Spring was making the particle heavier.
After correcting for both of these factors, they found that the particle got suddenly heavier multiple times during the day, at the same times. This was because a train runs near the detector whose electromagnetic fields were disturbing the experiment.
This is how precise the experiment was.”

LEP Pulls



Measurement
Fit
$|O^{meas}-O^{fit}|/\sigma^{meas}$
0 1 2 3
$\Delta\alpha_{had}^{(5)}(m_Z)$ 0.02758 ± 0.00035 0.02766
m_Z [GeV] 91.1875 ± 0.0021 91.1874
Γ_Z [GeV] 2.4952 ± 0.0023 2.4957
σ_{had}^{0} [nb] 41.540 ± 0.037 41.477
R_l 20.767 ± 0.025 20.744
$A_{fb}^{0,l}$ 0.01714 ± 0.00095 0.01640
$A_l(P_\tau)$ 0.1465 ± 0.0032 0.1479
R_b 0.21629 ± 0.00066 0.21585
R_c 0.1721 ± 0.0030 0.1722
$A_{fb}^{0,b}$ 0.0992 ± 0.0016 0.1037
$A_{fb}^{0,c}$ 0.0707 ± 0.0035 0.0741
A_b 0.923 ± 0.020 0.935
A_c 0.670 ± 0.027 0.668
A_l(SLD) 0.1513 ± 0.0021 0.1479
$\sin^2\theta_{eff}^{lept}(Q_{fb})$ 0.2324 ± 0.0012 0.2314
m_W [GeV] 80.392 ± 0.029 80.371
Γ_W [GeV] 2.147 ± 0.060 2.091
m_t [GeV] 171.4 ± 2.1 171.7
0 1 2 3